ON CENTRALLY (QUASI-)MORPHIC MODULES

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ABSTRACT. This paper investigates centrally (quasi-)morphic modules as a natural generalization of centrally morphic rings. An R-module M is defined as *centrally quasi-morphic* if, for every endomorphism $f \in \mathrm{End}_R(M)$, there exist central elements $g, h \in \mathrm{End}_R(M)$ such that $\ker(f) = \mathrm{Im}(g)$ and $\mathrm{Im}(f) = \ker(h)$. Furthermore, M is termed *centrally morphic* if $g = h$ satisfies the given conditions. A ring R is said to be *right (left) centrally (quasi-)morphic* if R_R $({}_RR)$ is centrally (quasi-)morphic. We prove that R is right centrally morphic if and only if it is right centrally quasi-morphic. Furthermore, modules whose endomorphism rings are regular (resp., unit-regular or strongly regular) are termed *endoregular* (resp. *unit-endoregular* or *strongly endoregular*). While it is known that every endoregular module is quasi-morphic and every unit-endoregular module is morphic, we show that every strongly endoregular module is centrally morphic. Finally, these relationships are summarized in a diagram, and we provide illustrative examples demonstrating that these implications are not generally reversible.

$$
\begin{tikzcd}
\text{strongly endoregular} \arrow[r, shift left] \arrow[d, shift right] & \text{unit endoregular} \arrow[l, shift left, "/" marking] \arrow[r, shift left] \arrow[d, shift right] & \text{endoregular} \arrow[l, shift left, "/" marking] \arrow[d, shift right] \\
\text{centrally morphic} \arrow[u, shift right, "/" marking] \arrow[r, shift left] \arrow[dr] & \text{morphic} \arrow[u, shift right, "/" marking] \arrow[l, shift left, "/" marking] \arrow[r, shift left] & \text{quasi-morphic} \arrow[u, shift right, "/" marking] \arrow[l, shift left, "/" marking] \arrow[dl, shift left, "/" marking] \\
& \text{centrally quasi-morphic.} \arrow[ur, shift left] &
\end{tikzcd}
$$

We demonstrate that for image-projective modules, the notions of centrally morphic and centrally quasi-morphic coincide, and we further establish that every centrally quasi-morphic module is abelian. Regarding semisimple modules, we prove that a module M_R is centrally (quasi-)morphic if and only if it is strongly endoregular, which is equivalent to the condition that every homogeneous component of M_R has length 1. Additionally, we provide a complete characterization of centrally (quasi-)morphic modules that are direct sums of cyclic modules over principal ideal domains. Finally, we establish various structural properties of modules satisfying these conditions.

1. MAIN RESULTS

We recall that a ring R is called *right centrally morphic* if for each $a \in R$, there exists $b \in \mathrm{Cent}(R)$ such that $\mathrm{r.ann}_R(a) = bR$ and $aR = \mathrm{r.ann}_R(b)$. Left centrally morphic rings are defined similarly. A left and right centrally morphic ring is called *centrally morphic*. Clearly every right centrally morphic ring is right morphic, while the converse does not hold generally. Strongly regular rings and commutative morphic rings are centrally morphic, however, there exists a centrally morphic ring that is neither strongly regular nor commutative (for more details, see [5, Examples 22]). We begin with the following proposition from [1].

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Proposition 1.1. [1, Proposition 3.1] *The following statements are equivalent for a ring R:*
(a) *R is left centrally morphic;*
(b) *For every $a \in R$, there exist elements $b, c \in \mathrm{Cent}(R)$ such that $\mathrm{l.ann}_R(a) = Rb$ and $Ra = \mathrm{l.ann}_R(c)$.*

In this paper, we extend the centrally morphic notion from rings to modules. Let M be an R-module. We say that M_R is *centrally quasi-morphic* if for every $f \in S$ there exist $g, h \in \mathrm{Cent}(S)$ such that $\mathrm{Im}f = \mathrm{Ker}h$ and $\mathrm{Ker}f = \mathrm{Im}g$. Moreover, M_R is said to be *centrally morphic* if we can choose $g = h$ in the above definition. Equivalently, an R-module M is centrally quasi-morphic if for any $f \in S$, there exist $g, h \in \mathrm{Cent}(S)$ such that the sequence $M_R \xrightarrow{g} M_R \xrightarrow{f} M_R \xrightarrow{h} M_R$ is exact. A ring R is right (left) centrally (quasi-)morphic if and only if R_R (${}_RR$) is centrally (quasi-)morphic. Besides, by Proposition 1.1, R_R is centrally morphic if and only if R_R is centrally quasi-morphic. Clearly every centrally morphic module is centrally quasi-morphic.
In the following we extend Proposition 1.1, by showing that for image-projective modules, the notions of centrally quasi-morphic and centrally morphic coincide. Recall that M_R is called *image-projective* if $\alpha(M) \subseteq \beta(M)$ where $\alpha, \beta \in S$, then $\alpha \in \beta S$ [7, Section 4]. It is clear that every (quasi-)projective module is image-projective. Besides, M_R generates its kernels whenever M_R generates $\mathrm{Ker}\alpha$ for every $\alpha \in S$.

Proposition 1.2. *The following statements are equivalent for an R-module M:*
(a) *M_R is image-projective and centrally morphic;*
(b) *M_R is image-projective and centrally quasi-morphic;*
(c) *S is right centrally morphic and M_R generates its kernels.*

Corollary 1.3. *Let M be a projective R-module. Then M_R is centrally morphic if and only if it is centrally quasi-morphic.*

Recall that a ring R is called *abelian* if each of its idempotents is central. Moreover, an R-module M is said to be *abelian* if S is an abelian ring. It is easy to check that M_R is abelian if and only if every direct summand of M_R is fully invariant. It has been proved that every left centrally morphic ring is abelian [5, Lemma 21]. In the next proposition, we prove that centrally quasi-morphic modules are abelian.

Proposition 1.4. *Let M be a centrally quasi-morphic R-module. The following statements hold:*
(a) *For every $f \in S$, $\mathrm{Im}f$ and $\mathrm{Ker}f$ are fully invariant submodules of M_R;*
(b) *M_R is abelian.*

Proposition 1.5. *Let R be a ring. The following statements hold:*
(a) *Let M be an R module and $MI = 0$ where I is an ideal of R. Then $M_{R/I}$ is centrally (quasi-)morphic if and only if M_R has the property;*
(b) *Every direct summand of a centrally (quasi-)morphic module has the property;*
(c) *Let $\{M_i\}_{i \in I}$ be a family of R-modules. Then $\oplus_{i \in I} M_i$ is centrally (quasi-)morphic if and only if each M_i is centrally (quasi-)morphic and $\mathrm{Hom}_R(M_i, M_j) = 0$ where $i \neq j$ and $i, j \in I$*

Corollary 1.6. *Let F be a free R-module. Then F_R is centrally (quasi-)morphic if and only if $F \simeq R$ and R is right centrally morphic.*

Proposition 1.7. *Let V be a vector space over a field F. The following statements hold:*
(a) *V_F is quasi-morphic;*
(b) *V_F is morphic if and only if it has finite dimension;*
(c) *V_F is centrally (quasi-)morphic if and only if $\dim(V_F) = 1$.*

In the next two results from [2], the condition when the direct sum of cyclic abelian groups is (quasi-) morphic has been completely characterized. We are also interested in providing the condition when the direct sum of cyclic abelian groups is centrally (quasi-) morphic.

Theorem 1.8. [2, Corollary 3.4] *Let G be an abelian group which is a direct sum of cyclic groups. Then $G_{\mathbb{Z}}$ is morphic if and only if $G_{\mathbb{Z}}$ is torsion and every its p-primary component has the form $\mathbb{Z}_{p^n}^{(k)}$ where $k \geq 0$ and $n \geq 0$.*

Theorem 1.9. [2, Theorem 3.8] *Let G be a quasi-morphic abelian group which is a direct sum of cyclic groups. Then $G_{\mathbb{Z}}$ is torsion and every its p-primary component has the form $\oplus_{i=1}^{t} \mathbb{Z}_{p^{n_i}}^{(\Lambda_i)}$*

Theorem 1.10. *Let R be a principal ideal domain which is not a field and M_R be a direct sum of cyclic R-modules. Then M_R is centrally (quasi-)morphic if and only if M_R is torsion and each of its p-primary component has the form R/p^nR where p is a prime element of R.*

Corollary 1.11. *Let G be an abelian group which is a direct sum of cyclic groups. Then $G_{\mathbb{Z}}$ is centrally (quasi-)morphic if and only if $G_{\mathbb{Z}}$ is torsion and each p-primary component has the form $\mathbb{Z}_{p^n}$ where $n \in \mathbb{N}$.*

We were not able to find a centrally quasi-morphic module which is not centrally morphic. So we end this section by asking the next question.

Question 1.12. Is every centrally quasi-morphic module centrally morphic?

In 2013, Lee, Rizvi, and Roman [3], investigated modules whose endomorphism rings are regular, which are called *endoregular*. Moreover, Zhang and Lee, in [4], considered the case in which the endomorphism ring of a module is a unit-regular ring. These modules are called *unit endoregular*. Regarding these two cases, we are interested in considering modules whose endomorphism ring is strongly regular, which we shall call them *strongly endoregular*, and are studying the relation between these modules and the centrally (quasi-) morphic modules. Clearly, a ring R is strongly regular whenever R_R is strongly endoregular. Endoregular modules are quasi-morphic and every unit-endoregular module is morphic [2, Corollary 2.14]. Strongly regular rings are centrally morphic [5, Example 22(1)]. We extend this result for modules by showing that every strongly endoregular module is centrally morphic. First, we obtain several properties of strongly endoregular modules in the next theorem.

Theorem 1.13. *Let M be an R-module. The following statements are equivalent:*
(a) *M_R is strongly endoregular;*
(b) *M_R is abelian and endoregular;*
(c) *For any $f \in S$, $\mathrm{Im} f = \mathrm{Im} f^2$ and $\mathrm{Ker} f = \mathrm{Ker} f^2$;*
(d) *For every $f \in S$, $\mathrm{Im} f \oplus \mathrm{Ker} f = M$;*
(e) *For every $f \in S$, there exists an (central) idempotent element $e \in S$ such that the sequence $M \xrightarrow{e} M \xrightarrow{f} M \xrightarrow{e} M$ is exact;*

(f) *For every* $f \in S$*, there exist idempotent elements* $e_1, e_2 \in \mathrm{Cent}(S)$ *such that the sequence* $M \xrightarrow{e_1} M \xrightarrow{f} M \xrightarrow{e_2} M$ *is exact;*
(g) *For submodules* N *and* K *of* M_R*,* $M/N \simeq K$ *implies that* $M = N \oplus K$*.*

Corollary 1.14. *Every strongly endoregular module is centrally morphic.*

In the following, we state conditions under which the converse of Corollary 1.14 does hold. This can be compared with the next result of [4].

It is well known that every semisimple module is endoregular. While it is routine to see that semisimple unit endoregular modules are precisely modules whose homogenous components are finitely generated. In the next, semisimple strongly endoregular modules are characterized.

Lemma 1.15. *Let* R *be a ring and* M *be an indecomposable* R*-module. The following statements are equivalent:*
(a) M_R *is strongly endoregular;*
(b) M_R *is (unit) endoregular;*
(c) S *is a division ring.*

Theorem 1.16. *Let* R *be a ring and* M *be an* R*-module such that* $M = \oplus_{i \in I} M_i$ *where each* M_i *is an indecomposable* R*-module. Then* M_R *is strongly endoregular if and only if* $S := \mathrm{End}_R(M)$ *is a direct product of division rings.*

Proposition 1.17. *Let* M *be a semisimple* R*-module. The following statements are equivalent:*
(a) M_R *is strongly endoregular;*
(b) M_R *is centrally (quasi-)morphic;*
(c) *Every homogeneous component of* M_R *is of length 1.*

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